MULTIPLE INTEGRALS - TASKS (I PART)

Double integrals-determination of the limits of integration

The first thing we face with the double integrals is the determination of the limits of integration.

For almost every task we have to draw the picture, so, it is best to as to remind the basic graphics.

(You have a file on the site)

We have two basic types of areas of integration:

1)

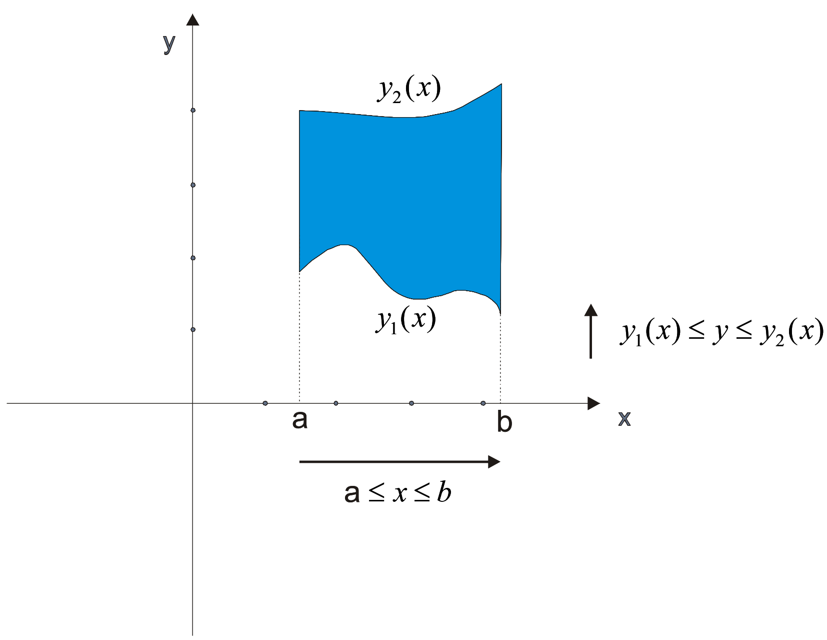
If the area of integration D is bounded in left and right sides with lines x = a and x = b (say a < b),

the top and bottom with continuous functions $y_1(x)$ and $y_2(x)$ where $y_1(x) \le y_2(x)$, then we have:

$$\begin{cases} a \le x \le b \\ y_1(x) \le y \le y_2(x) \end{cases}$$

and :
$$\iint_D z(x,y)dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} z(x,y)dy$$

Let's look at the picture:



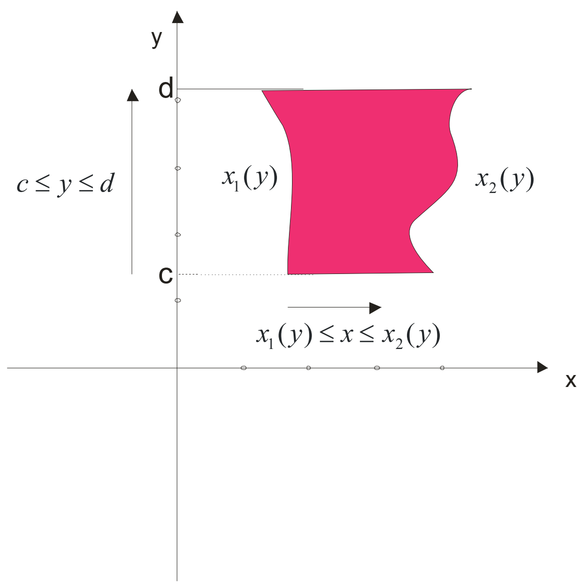
This is a common situation, when we first solve integral “by dy” where x will be treated as a constant, and then solve an ordinary integral “by dx”

2)

In the second situation, the area of integration D is bounded from below and above with lines y = c and y = d ,

where c<d, and on the left and right functions are expressed through $x_1(y)$ and $x_2(y)$, where $x_1(y) \le x_2(y)$

Let's look at the picture:



So:

Consider the area given with :

$$\begin{cases} x_1(y) \le x \le x_2(y) \\ c \le y \le d \end{cases}$$

then: $\iint\limits_D z(x,y)dxdy = \int\limits_c^d dy \int\limits_{x_1(y)}^{x_2(y)} z(x,y)dx$

Here is the first integral "by dx" and then the integral by dy .

Which type you use depends on the specific situation .

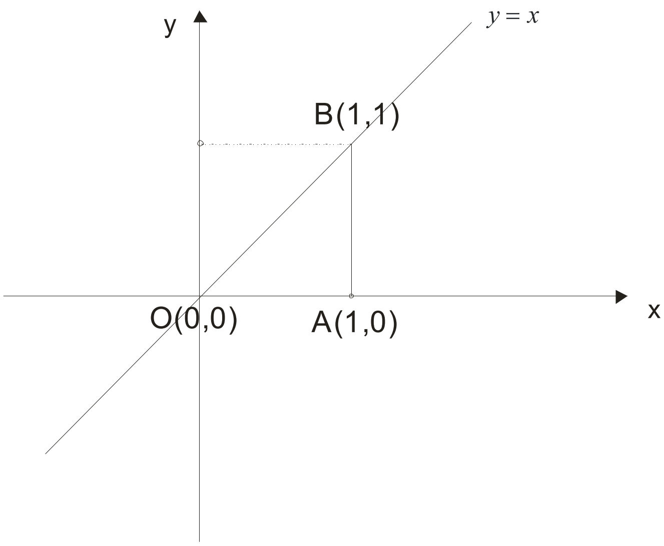
Draw a picture, find sections and then go to work ...

Example 1.

Determine the limits of integration of double integral $\iint_D z(x,y)dxdy$ for each possible order of integration if area D is triangle with vertices O(0,0); A(1,0) and B (1,1)

Solution:

First, we draw a picture :



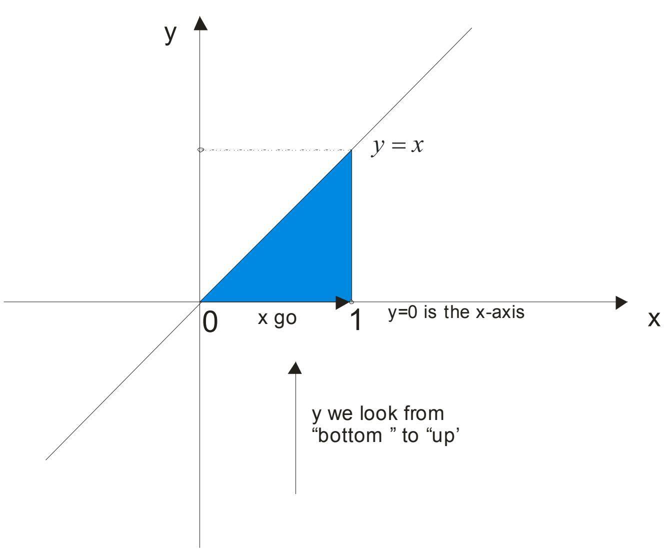
Line through points A and B, we will find as an equation of the line through two given points.

(if you do not know by heart to determine)

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

So : $y-0=\frac{1-0}{1-0}(x-0)\rightarrow \boxed{y=x}$

Come to do a border for the first order, look at the picture:



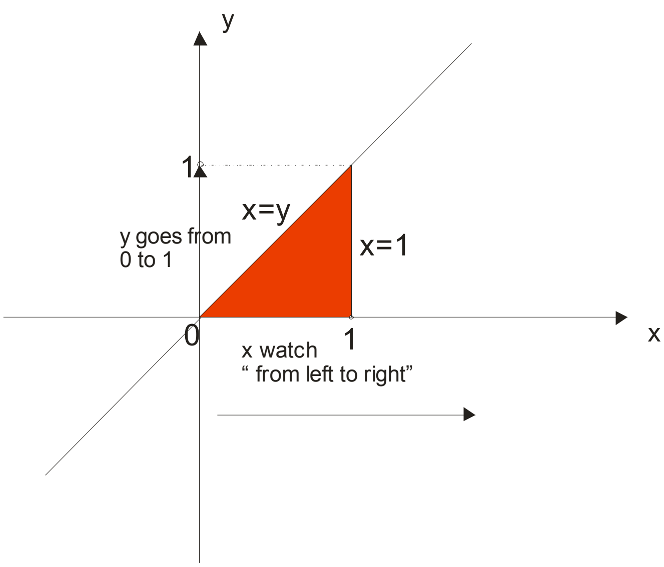
To limit by x we look from left to right. First we come to zero, and then to 1. So : $0 \le x \le 1$.

When we look at y, first find the x-axis, and we know that it is y = 0. On top is line y=x, so : $0 \le y \le x$

Area D is : $\begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$

Here we solve the integral given by the limits: $\iint_D z(x,y)dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} z(x,y)dy = \int_0^1 dx \int_0^x z(x,y)dy$

For the second order of integration we have:



Now y goes from 0 to 1 looking from the bottom up, so : $0 \le y \le 1$.

For x border look from left to right. First, we find the line x=y and then line x=1, so: $y \le x \le 1$

Area Dis now : $\begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases}$

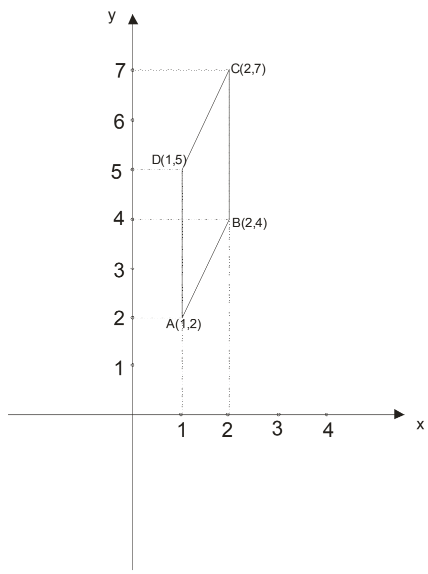
We will do : $\iint_D z(x,y)dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} z(x,y)dx = \int_0^1 dy \int_y^1 z(x,y)dx$

Example 2.

Determine the limits of integration of double integral $\iint_D z(x,y)dxdy$ for each possible order of integration if area D is a parallelogram with vertices A(1,2); B(2,4); C(2,7) and D(1,5)

<u>Solution:</u>

We draw a picture:

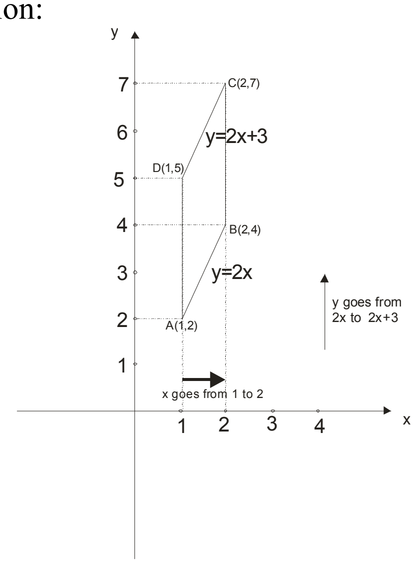


We need equations of lines through the AB and CD. We will use $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ and get:

AB: $y = 2x$

CD: $y = 2x+3$

Now we consider the first order of integration:



Area D is: $\begin{cases} 1 \le x \le 2 \\ 2x \le y \le 2x+3 \end{cases}$ and:

$$\iint_D z(x,y)dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} z(x,y)dy = \int_1^2 dx \int_{2x}^{2x+3} z(x,y)dy$$

And this would be an easier way to solve ...

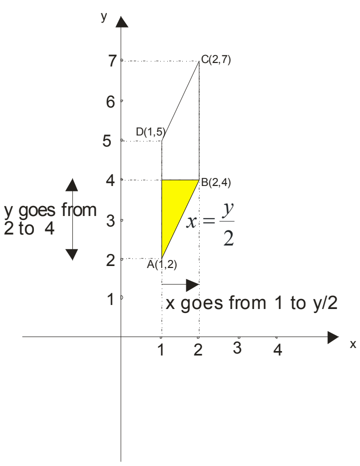
For the second order of integration the situation would be a little heavier.

Of course, first we will equation lines AB and CD expressed in terms by x.

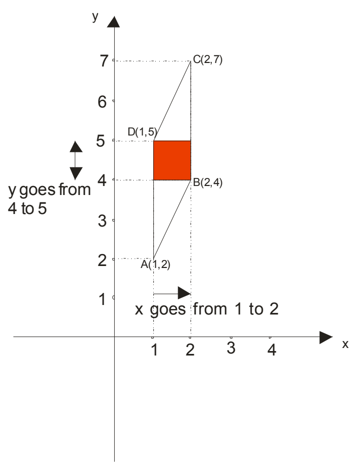
AB: $y = 2x \to x = \frac{y}{2}$

CD: $y = 2x+3 \to x = \frac{y-3}{2}$

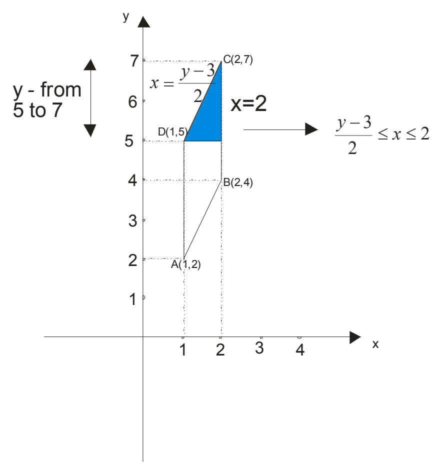
Pictures:



picture 1.



picture 2.



picture 3.

We have to share the area of integration into three parts:

$$D_1 : \begin{cases} 2 \le y \le 4 \\ 1 \le x \le \frac{y}{2} \end{cases} \qquad D_2 : \begin{cases} 4 \le y \le 5 \\ 1 \le x \le 2 \end{cases} \qquad D_3 : \begin{cases} 5 \le y \le 7 \\ \frac{y-3}{2} \le x \le 2 \end{cases}$$

Integral is now:

$$\iint\limits_D z(x,y)dxdy = \int\limits_c^d dy \int\limits_{x_1(y)}^{x_2(y)} z(x,y)dx = \int\limits_2^4 dy \int\limits_1^{\frac{y}{2}} z(x,y)dx + \int\limits_4^5 dy \int\limits_1^2 z(x,y)dx + \int\limits_5^7 dy \int\limits_{\frac{y-3}{2}}^2 z(x,y)dx$$

Example 3.

Determine the limits of integration of double integral $\iint_D z(x,y)dxdy$ for each possible order of integration if area D is limited with lines $y = x$ and $y = \sqrt{4x - x^2}$

Solution:

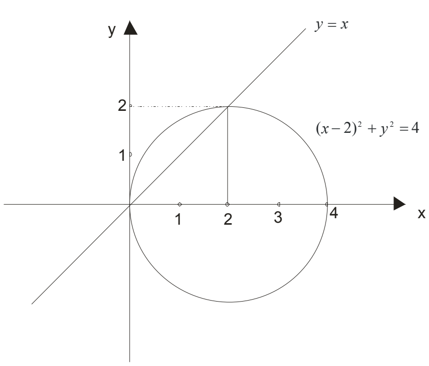
Curve $y = \sqrt{4x - x^2}$ is a circle but we must first sort out ..

$$
\begin{aligned}
&y = \sqrt{4x - x^2} \text{...................} / ()^2 \\
&y^2 = 4x - x^2 \\
&x^2 - 4x + y^2 = 0 \\
&\boxed{x^2 - 4x + 4} - 4 + y^2 = 0 \\
&(x-2)^2 + y^2 = 4
\end{aligned}
$$

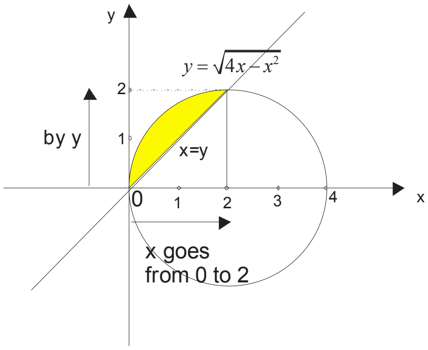
Intersection we will get by solving the equation system:

$$
\begin{aligned}
&y = \sqrt{4x - x^2} \wedge y = x \\
&x = \sqrt{4x - x^2} \text{..................} ()^2 \\
&x^2 = 4x - x^2 \\
&2x^2 - 4x = 0 \\
&2x(x-2) = 0 \rightarrow x = 0 \vee x = 2 \\
&x = 0 \rightarrow y = 0 \\
&x = 2 \rightarrow y = 2
\end{aligned}
$$

Picture will be:



The first order of integration will be:



$$D:\begin{cases} 0\le x\le 2 \\ x\le y\le \sqrt{4x-x^2}\end{cases} \text{ and integral will be: } \iint_D z(x,y)dxdy=\int_a^b dx\int_{y_1(x)}^{y_2(x)} z(x,y)dy=\int_0^2 dx\int_x^{\sqrt{4x-x^2}} z(x,y)dy$$

For the second order of integration, as in previous examples, we have a little more work ….

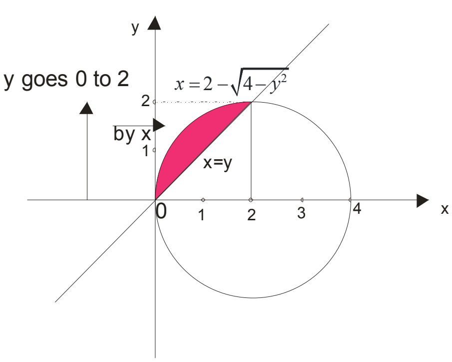
First, to express x from $y=\sqrt{4x-x^2}$:

$$y=\sqrt{4x-x^2}$$
$$y^2=4x-x^2$$
$$x^2-4x+y^2=0$$

This now solve a quadratic equation:

$$x^2-4x+y^2=0$$
$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{4\pm\sqrt{16-4y^2}}{2}=\frac{4\pm2\sqrt{4-y^2}}{2}$$
$$x_{1,2}=2\pm\sqrt{4-y^2}\to we\ need\to \boxed{x=2-\sqrt{4-y^2}}$$

Now picture :



$$D:\begin{cases} 0\le y\le 2 \\ 2-\sqrt{4-y^2}\le x\le y\end{cases} \text{ and integral is : } \iint_D z(x,y)dxdy=\int_c^d dy\int_{x_1(y)}^{x_2(y)} z(x,y)dx=\int_0^2 dy\int_{2-\sqrt{4-y^2}}^{y} z(x,y)dx$$

Example 4.

Change the order of integration in the integral: $\int_0^1 dy \int_y^{\sqrt{y}} z(x,y)dx$

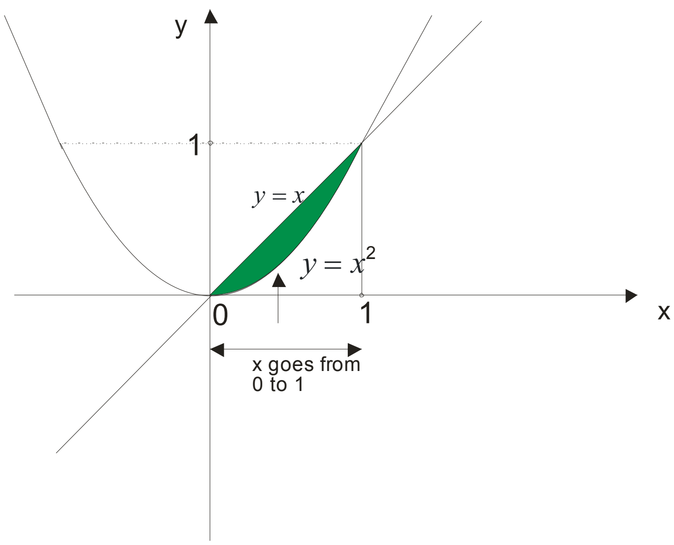
Solution:

From integral $\int_0^1 dy \int_y^{\sqrt{y}} z(x,y)dx$ we can see that:

$$D: \begin{cases} 0 \le y \le 1 \\ y \le x \le \sqrt{y} \end{cases}$$

From $x = y \to y = x$ and $\sqrt{y} = x \to y = x^2$

Picture:



Area D is now $D: \begin{cases} 0 \le x \le 1 \\ x^2 \le y \le x \end{cases}$ and integral is: $\int_0^1 dy \int_y^{\sqrt{y}} z(x,y)dx = \int_0^1 dx \int_{x^2}^{x} z(x,y)dy$

Example 5.

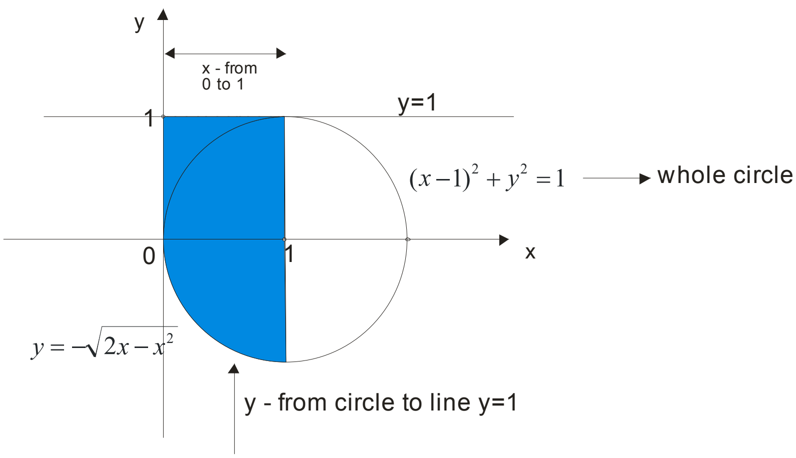
Change the order of integration in the integral: $\int_0^1 dx \int_{-\sqrt{2x-x^2}}^{1} z(x,y)dy$

Solution:

Let's “get” a circle and draw a picture:

$$y=-\sqrt{2x-x^2}\,..............()^2$$
$$y^2=2x-x^2$$
$$x^2-2x+y^2=0$$
$$\boxed{x^2-2x+1}-1+y^2=0$$
$$(x-1)^2+y^2=1$$

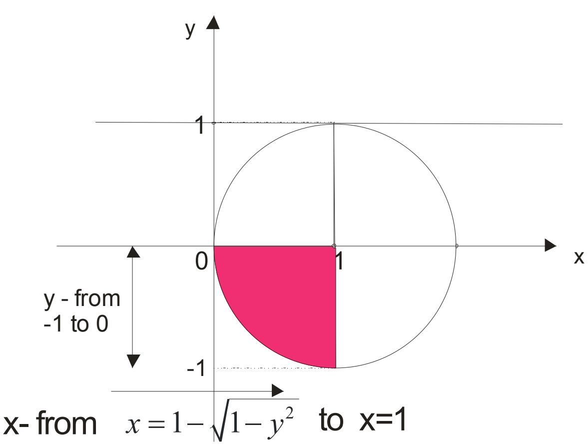
Picture:



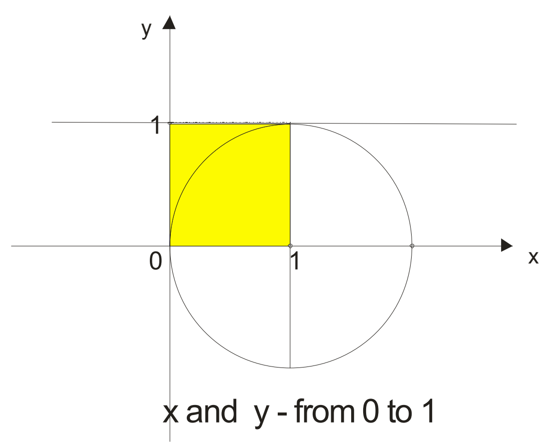
Now, we must express x from the circle :

$$y=-\sqrt{2x-x^2}\,..............()^2$$
$$y^2=2x-x^2$$
$$x^2-2x+y^2=0$$
$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4-4y^2}}{2}=\frac{2\pm2\sqrt{1-y^2}}{2}=1\pm\sqrt{1-y^2}\rightarrow\boxed{x=1-\sqrt{1-y^2}}$$

We have to divide the area into two parts:



Area D_1



Area D_2

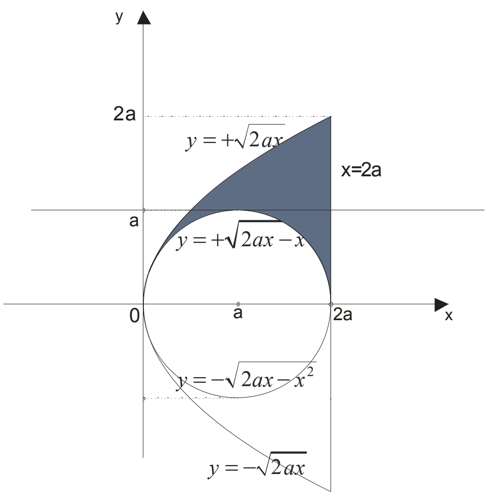
Integral will be:

$$\int_{0}^{1} dx \int_{-\sqrt{2x-x^2}}^{1} z(x,y)dy = \int_{-1}^{0} dy \int_{1-\sqrt{1-y^2}}^{1} z(x,y)dx + \int_{0}^{1} dy \int_{0}^{1} z(x,y)dx$$

Example 6.

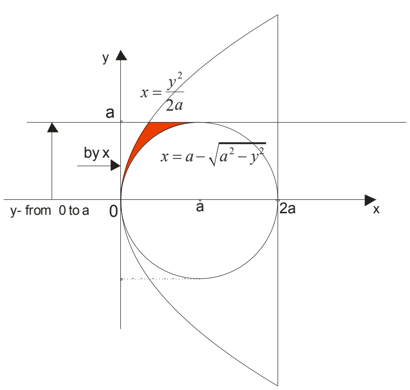
Change the order of integration in the integral: $\int_{0}^{2a} dy \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} z(x,y)dx, \qquad a > 0$

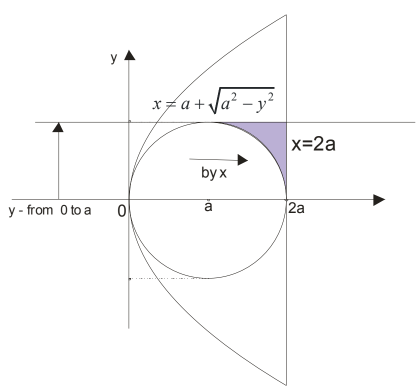
Solution:



Painted area is our area of integration ..

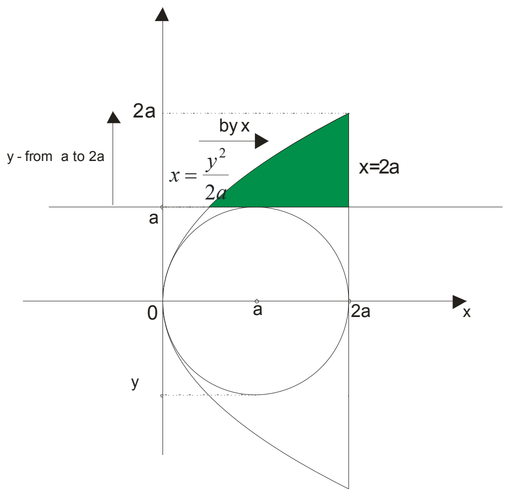
To change the order of integration we "see" three areas:





$$D_1 : \begin{cases} 0 \le y \le a \\ \dfrac{y^2}{2a} \le x \le a - \sqrt{a^2 - y^2} \end{cases} \qquad D_2 : \begin{cases} 0 \le y \le a \\ a + \sqrt{a^2 - y^2} \le x \le 2a \end{cases}$$

The third part would be:



$$D_3 : \begin{cases} a \le y \le 2a \\ \frac{y^2}{2a} \le x \le 2a \end{cases}$$

Example 7.

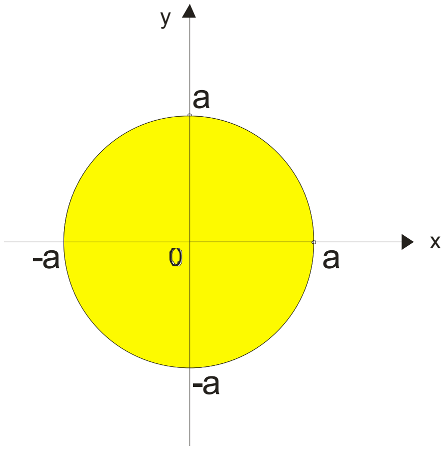
In dual integral $\iint_D z(x, y)dxdy$ switch to polar coordinates where area D is the circle $x^2 + y^2 \le a^2$

Solution:

To recall the first to go over to polar coordinates (J is the Jacobian)

$$\begin{cases} \text{x} = \text{r} \cos\varphi \\ \text{y} = \text{r} \sin\varphi \quad \text{then :} \quad \iint_D z(x, y)dxdy = \iint_{D`} z(r\cos\varphi, r\sin\varphi)|J|drd\varphi = \int_{\varphi_1}^{\varphi_2} d\varphi \int_0^r z(r\cos\varphi, r\sin\varphi)rdr \\ |J| = \text{r} \end{cases}$$

Draw picture and go to polar coordinates:



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \rightarrow |J| = r \quad \text{this replace in } x^2 + y^2 \le a^2$$

$$x^2 + y^2 = a^2$$
$$(r\cos\varphi)^2 + (r\sin\varphi)^2 = a^2$$
$$r^2(\cos^2\varphi + \sin^2\varphi) = a^2 \quad \text{we know that} \quad \cos^2\varphi + \sin^2\varphi = 1$$
$$r^2 = a^2 \rightarrow r = a$$

So, r goes from 0 to a.

We need whole circuit, and it is clear that $0 \le \varphi \le 2\pi$

So: $D` = \begin{cases} 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases}$ and :

$$\iint_D z(x,y)dxdy = \iint_{D`} z(r\cos\varphi, r\sin\varphi)|J|drd\varphi = \int_0^{2\pi} d\varphi \int_0^a z(r\cos\varphi, r\sin\varphi)rdr$$

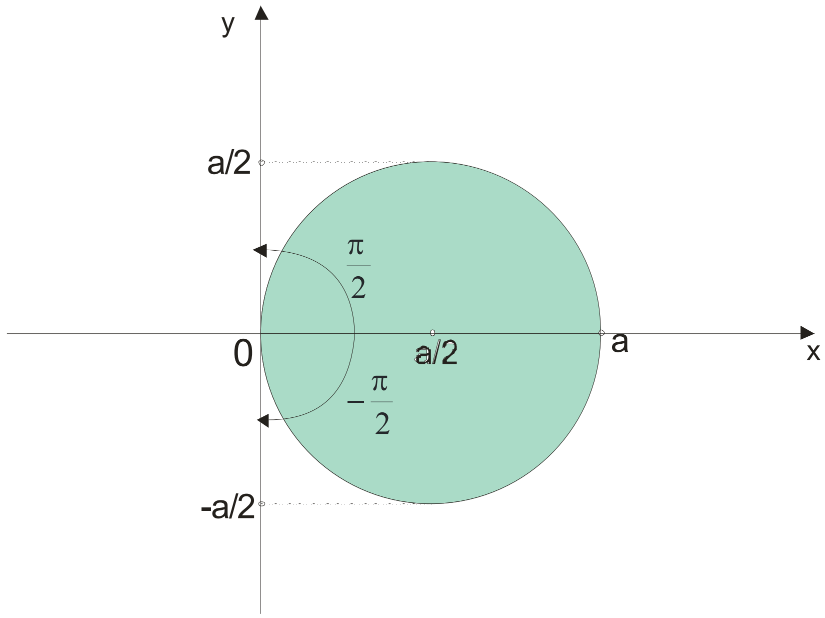
Example 8.

In dual integral $\iint_D z(x,y)dxdy$ switch to polar coordinates where area D is the circle $x^2 + y^2 \le ax$

Solution :

To pack a circle first, then we draw a picture:

$$x^2 + y^2 = ax$$
$$x^2 - ax + y^2 = 0$$
$$x^2 - ax + \frac{a^2}{4} - \frac{a^2}{4} + y^2 = 0$$
$$\boxed{(x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}}$$



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \rightarrow |J| = r$$

$$x^2 + y^2 = ax$$
$$(r\cos\varphi)^2 + (r\sin\varphi)^2 = ar\cos\varphi$$
$$r^2(\cos^2\varphi + \sin^2\varphi) = ar\cos\varphi$$
$$r^2 = ar\cos\varphi \rightarrow r = a\cos\varphi$$

So: $0 \le r \le a\cos\varphi$

We must be careful in terms of angle, because now we do not need the whole circle, but (see picture) : $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

So we have : D\`= $\begin{cases} 0 \le r \le a\cos\varphi \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \end{cases}$ and :

$$\iint_D z(x,y)dxdy = \iint_{D`} z(r\cos\varphi, r\sin\varphi)|J|drd\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a\cos\varphi} z(r\cos\varphi, r\sin\varphi)rdr$$